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ON TWISTED HOMOGENEOUS RACKS OF TYPE D

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Abstract. We develop some techniques to check when a twisted homogeneous rack of class (L, t, θ) is of type D. Then we apply the obtained results to the cases $L = \mathbb{A}_n$, $n \geq 5$, or L a sporadic group.

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1. Introduction

A *rack* is a pair $(X, \triangleright)$ where X is a non-empty set and $\triangleright : X \times X \to X$ is an operation such that the map $\varphi_x = x \triangleright \_\_$ is bijective for any $x \in X$, and $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ for all $x, y, z \in X$. For instance, a conjugacy class in a group is a rack with $\triangleright$ given by conjugation. Racks appear naturally in the classification problem of finite-dimensional pointed Hopf algebras, see [AG]. In this sense, our general aim is to solve the following question. We omit the definitions of *cocycle over a rack* and *Nichols algebra* $\mathfrak{B}(X, \mathbf{q})$ because these are not needed in the present paper, an omission justified by Theorem 1.1 below.

Question 1. *For any finite rack X, and for any cocycle $\mathbf{q}$, determine if* $\dim \mathfrak{B}(X, \mathbf{q}) < \infty$.

A rack X is *of type D* if there exists a decomposable subrack $Y = R \coprod S$ of X such that

$$r \triangleright (s \triangleright (r \triangleright s)) \neq s, \quad \text{for some } r \in R, s \in S. \tag{1}$$

The reduction of certain problems in the classification of finite-dimensional pointed Hopf algebras to the study of racks of type D is given by the following result, a reinterpretation of [HS, Thm. 8.6] in turn a consequence of [AHS].

Theorem 1.1. [AFGV1, Th. 3.6] *If X is a finite rack of type D, then X collapses, that is* $\dim \mathfrak{B}(X, \mathbf{q}) = \infty$ *for any* $\mathbf{q}$. □

Therefore, it is very important to determine all simple racks of type D. Indeed, any indecomposable rack Z admits a rack epimorphism $\pi : Z \to X$ with X simple; and if X is of type D, then Z is of type D.

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Now, finite simple racks have been classified in [AG, 3.9, 3.12], see also [J]. Succinctly, any simple rack belongs to one of the following classes:

(1) Simple affine racks.
(2) Non-trivial (twisted) conjugacy classes in simple groups.
(3) *Simple twisted homogeneous racks* of class (L, t, θ); these are twisted conjugacy classes corresponding to (G, u), where

- $G = L^t$, with L a finite simple non-abelian group, $t > 1$ and $\theta \in \mathrm{Aut}(L)$,
- $u \in \mathrm{Aut}(L^t)$ acts by

$$u(\ell_1, \dots, \ell_t) = (\theta(\ell_t), \ell_1, \dots, \ell_{t-1}), \quad \ell_1, \dots, \ell_t \in L.$$

See Subsection 3.2 below.

The goal of the present paper is to study when a simple twisted homogeneous rack, that is case (3), is of type D. We first show that twisted homogeneous racks (THR, for short) of class (L, t, θ) are parameterized by twisted conjugacy classes in L with respect to θ, see Proposition 3.3. For example, assume that $L = \mathbb{A}_n$, $n \geq 5$, and $\theta = \iota_{(1\ 2)}$ (conjugation in $\mathbb{S}_n$). Then the THR of type $(\mathbb{A}_n, t, \iota_{(1\ 2)})$ are parameterized by the conjugacy classes in $\mathbb{S}_n$ not contained in $\mathbb{A}_n$; this explain the notation in Table 2. Then we develop some techniques to check that a twisted homogeneous rack is of type D, see Section 3; neither of these results requires the simplicity of L. Our main results are proved in Section 4:

Theorem 1.2. *Let L be $\mathbb{A}_n$, $n \geq 5$, $\theta \in \mathrm{Aut}(L)$, $t \geq 2$ and $\ell \in L$. If $\mathcal{C}_\ell$ is a twisted homogeneous rack of class (L, t, θ) not listed in Tables 1, 2, then $\mathcal{C}_\ell$ is of type D.*

Theorem 1.3. *Let L be a sporadic group, $\theta = \mathrm{id}$, $t \geq 2$ and $\ell \in L$. If $\mathcal{C}_\ell$ is a twisted homogeneous rack of class (L, t, θ) not listed in Table 3, then $\mathcal{C}_\ell$ is of type D.*

The case: L a sporadic group, $\theta \neq \mathrm{id}$, will be treated in another paper. See however a preliminary discussion in Subsection 4.4.

2. Preliminaries

2.1. **Glossary.** If $t, n \in \mathbb{N}$, then (t, n) denotes their highest common divisor. If G is a group and $x \in G$, then $\langle x \rangle$ denotes the cyclic subgroup generated by x; ι_x, the inner automorphism associated to x; $\mathcal{O}_x^G$, the conjugacy class of x, and $\mathrm{C}_G(x)$ its centralizer. We say that x is an involution if it has order 2. If $u \in \mathrm{Aut}(G)$, then G^u denotes the subgroup of points fixed by u. The trivial element of G is denoted by e. A decomposition of a rack X is a presentation as a disjoint union of two subracks $X = Y \coprod Z$. A rack is indecomposable if it admits no decomposition.

TABLE 1. THR $\mathcal{C}_\ell$ of type $(\mathbb{A}_n, t, \theta)$, $\theta = \mathrm{id}$, $t \geq 2$, $n \geq 5$, not known of type D. Those not of type D are in bold.

n	ℓ	Type of ℓ	t
any	e	(1^n)	odd, $(t, n!) = 1$
5		$(\mathbf{1^5})$	$\mathbf{2}$
5		(1^5)	4
6		$(\mathbf{1^6})$	$\mathbf{2}$
5	involution	$(1, 2^2)$	4, odd
6		$(1^2, 2^2)$	odd
8		(2^4)	odd
any	order 4	$(1^{r_1}, 2^{r_2}, 4^{r_4})$, $r_4 > 0$, $r_2 + r_4$ even	2

TABLE 2. THR $\mathcal{C}_\ell$ of type $(\mathbb{A}_n, t, \theta)$, $\theta = \iota_{(1\,2)}$, $t \geq 2$, $n \geq 5$, not known of type D.

n	Type of $\ell(1\ 2)$	t
any	$(1^{s_1}, 2^{s_2}, \dots, n^{s_n})$, $s_1 \leq 1$ and $s_2 = 0$ $s_h \geq 1$, for some h, $3 \leq h \leq n$	any
	$(1^{s_1}, 2^{s_2}, 4^{s_4})$, $s_1 \leq 2$ or $s_2 \geq 1$, $s_2 + s_4$ odd, $s_4 \geq 1$	2
5	$(1^3, 2)$	2, 4
6	$(1^4, 2)$	2
	(2^3)	2
7	$(1, 2^3)$	2, odd
8	$(1^2, 2^3)$	odd
10	(2^5)	odd

TABLE 3. THR $\mathcal{C}_\ell$ of type (L, t, θ), with L a sporadic group, $\theta = \mathrm{id}$, not known of type D.

sporadic group	Type of ℓ or class name of $\mathcal{O}_\ell^L$	t
any	1A	$(t, \vert L\vert) = 1$, t odd
	$\mathrm{ord}(\ell) = 4$	2
T, J_2, Fi_{22}, Fi_{23}, Co_2	2A	odd
B	2A, 2C	odd
Suz	6B, 6C	any

2.2. **Affine racks of type D.** Let A be a finite abelian group and $g \in \mathrm{Aut}(A)$. We denote by (A, g) the rack with underlying set A and rack multiplication $x \triangleright y := g(y) + (\mathrm{id} - g)(x)$, $x, y \in A$; this is a subrack of the group $A \rtimes \langle g \rangle$. Any rack isomorphic to some (A, g) is called *affine*.

Notice that (A, g) is indecomposable if and only if $\mathrm{id} - g$ is invertible. For, assume that $\mathrm{Im}(\mathrm{id} - g) \neq A$; since $x \triangleright y = g(y) + (\mathrm{id} - g)(x) = y + (\mathrm{id} - g)(x - y)$, the decomposition of A in cosets with respect to $\mathrm{Im}(\mathrm{id} - g)$ is a decomposition in the sense of racks. In fact, if Y is any coset, then $A \triangleright Y = Y$; thus, the union of any two different cosets is a decomposable subrack of A. The converse of the claim is [AFGV1, Remark 3.13].

For instance, consider the cyclic group $A = \mathbb{Z}/n$ and the automorphism g given by the inversion; the rack (A, g) is denoted $\mathcal{D}_n$ and called a dihedral rack. Thus, a family $(\mu_i)_{i \in \mathbb{Z}/n}$ of distinct elements of a rack X is isomorphic to $\mathcal{D}_n$ if $\mu_i \triangleright \mu_j = \mu_{2i-j}$ for all i, j.

Lemma 2.1. *If $m > 2$, then the dihedral rack $\mathcal{D}_{2m}$ is of type D.*

Proof. Since $\mathrm{id} - g = 2$ is not invertible, $X = \mathcal{D}_{2m} = \{1, 2, 3, ..., 2m\}$ is decomposable. Indeed, if $Y = \{1, 3, 5, ..., 2m - 1\}$, then $X \triangleright Y \subseteq Y$ and therefore $X = Y \coprod (X \backslash Y)$ is a decomposition. Let $r = 1 \in Y$ and $s = 2 \in X \backslash Y$. Then $r \triangleright (s \triangleright (r \triangleright s)) \neq s$, since $r \triangleright (s \triangleright (r \triangleright s)) = -2 + 2m$ and $m > 2$. $\square$

We now consider a generalization of this example. Let $k, t \in \mathbb{N}$, $k > 1$, $t > 2$ and consider the affine rack (A, g) where $A = (\mathbb{Z}/k)^{t-1}$ and

$$g(a_1, \dots, a_{t-1}) = \left(-\sum_{i=1}^{t-1} a_i, a_1, \dots, a_{t-2} \right). \tag{2}$$

Note that the case $t = 2$ corresponds to the dihedral rack.

Lemma 2.2. *If $(t, k) \neq 1$, and $k > 2$ when $t = 4$, then (A, g) is of type D.*

Proof. First we show that (A, g) contains at least two cosets with respect to $\mathrm{Im}(\mathrm{id} - g)$. Indeed, let $(a, 0, \dots, 0) \in A$ such that a does not belong to the image of $m_t : \mathbb{Z}/n \to \mathbb{Z}/n,\ x \mapsto tx$; such an a exists since $(t, k) \neq 1$. Then $(a, 0, \dots, 0) \notin \mathrm{Im}(\mathrm{id} - g)$, since otherwise there exists $(c_1, \dots, c_{t-1})$ such that

$$(a, 0, \dots, 0) = (\mathrm{id} - g)(c_1, \dots, c_{t-1}) = (c_1 + \sum_{i=1}^{t-1} c_i, c_2 - c_1, \dots, c_{t-1} - c_{t-2})$$

which implies that $c_i = c_j$ and $a = tc_1$ in $\mathbb{Z}/k$, a contradiction. Now, take $s = (0, \dots, 0)$ and $r = (1, 0, \dots, 0)$; then

$$r \triangleright (s \triangleright (r \triangleright s)) = \begin{cases} (2, -2, 2, -1, 0, \dots, 0), & t \geq 5; \\ (2, -2, 2), & t = 4; \\ (1, -2), & t = 3. \end{cases}$$

This is different to s, by hypothesis [1]. Thus (A, g) is of type D. $\square$

3. General techniques

3.1. **Twisted conjugacy classes.** Let G be a finite group and $u \in \mathrm{Aut}(G)$. G acts on itself by $y \rightharpoonup_u x = y\, x\, u(y^{-1})$, $x, y \in G$. Let $\mathcal{O}_x^{G,u}$ be the orbit of $x \in G$ by this action; we call it the *twisted conjugacy class* of x. Then $\mathcal{O}_x^{G,u}$ is a rack with operation

$$y \triangleright_u z = y\, u(z\, y^{-1}), \quad y, z \in \mathcal{O}_x^{G,u}. \tag{3}$$

The verification that this is a rack is straightforward, see [AG, Remark 3.8]. For instance, if $y = h \rightharpoonup_u x$ and $z = v \rightharpoonup_u x$, then $y \triangleright_u z = w \rightharpoonup_u x$, where $w = hxu(h^{-1}v)x^{-1}$. Of course, if $u = \mathrm{id}$, then this is just the rack structure on a conjugacy class.

The set of twisted conjugacy classes for u only depends on its class in $\mathrm{Out}(G)$. For, if $n \in G$ and $u' = u\iota_n$, where ι_n is the inner automorphism associated to n, then $R : (G, u) \to (G, u')$, $R(x) = x\, u(n^{-1})$, is an isomorphism of racks. Indeed, if $x, y \in G$, then

$$R(x \triangleright y) = xu(yx^{-1}n^{-1}) = xu(n^{-1})u(nyx^{-1}n^{-1}) = R(x) \triangleright' R(y).$$

Lemma 3.1. *Let $x \in G$ with $\mathrm{ord}(x) = 2m > 4$ and $u(x) = x^{-1}$. If $\langle x \rangle \subseteq \mathcal{O}_x^{G,u}$, then $\mathcal{O}_x^{G,u}$ is of type D.*

The hypothesis "$\langle x \rangle \subseteq \mathcal{O}_x^{G,u}$" is equivalent to the existence of $y \in G$ such that $y \rightharpoonup_u x = x^2$.

Proof. Since $x^i \triangleright_u x^j = x^i u(x^{j-i}) = x^{2i-j}$, $\langle x \rangle$ is a subrack of G isomorphic to $\mathcal{D}_{2m}$; hence Lemma 2.1 applies. $\square$

The following consequence of (3) helps the search of twisted homogeneous racks of type D.

Remark 3.2. If $x \in G^u$, then $\mathcal{O}_x^{G^u}$ is a subrack of $\mathcal{O}_x^{G,u}$.

Note that, if u is an involution, then

$$\mathcal{O}_x^{G,u} \cap G^u \neq \emptyset \iff \exists z \in G : zxz = u(x) \text{ and } \exists y \in G : z = u(y^{-1})y. \tag{4}$$

3.2. **Twisted homogeneous racks.** We fix a finite group L, $t \in \mathbb{N}$, $t > 1$, and $\theta \in \mathrm{Aut}(L)$. We consider in the rest of this section

- $G = L^t$,
- $u \in \mathrm{Aut}(G)$, $u(\ell_1, \dots, \ell_t) = (\theta(\ell_t), \ell_1, \dots, \ell_{t-1})$, $\ell_1, \dots, \ell_t \in L$.

[1] If $k = 2$ and $t = 4$, then (A, g) is not of type D.

The twisted conjugacy class of $(x_1, \dots, x_t) \in L^t$ will be called a *twisted homogeneous rack* of class (L, t, θ) and denoted $\mathcal{C}_{(x_1,\dots,x_t)}$, to avoid confusions. It is also useful to denote $\mathcal{C}_\ell := \mathcal{C}_{(e,\dots,e,\ell)}$, $\ell \in L$. We first describe the twisted homogeneous racks of class (L, t, θ).

Proposition 3.3. (i) *If* $(x_1, \dots, x_t) \in L^t$ *and* $\ell = x_t x_{t-1} \cdots x_2 x_1$, *then* $\mathcal{C}_{(x_1,\dots,x_t)} = \mathcal{C}_\ell$.

(ii) $\mathcal{C}_\ell = \mathcal{C}_k$ *iff* $k \in \mathcal{O}_\ell^{L,\theta}$; *hence*

$$\mathcal{C}_\ell = \{(x_1, \dots, x_t) \in L^t : \ x_t x_{t-1} \cdots x_2 x_1 \in \mathcal{O}_\ell^{L,\theta}\}.$$

(iii) *There exists a bijection* φ *between the set of twisted conjugacy classes of* L *and the set of twisted homogeneous racks of class* (L, t, θ), *given by* $\varphi(\mathcal{O}_\ell^{L,\theta}) = \mathcal{C}_\ell$.

(iv) $|\mathcal{C}_\ell| = |L|^{t-1} |\mathcal{O}_\ell^{L,\theta}|$.

Proof. (i). Let $u_j = (x_j \dots x_1)^{-1}$; then

$$\begin{aligned}(u_1, u_2, \dots, u_{t-1}, e) \rightharpoonup (x_1, \dots, x_t)&\\ &= (u_1 x_1, u_2 x_2 u_1^{-1}, \dots, u_{t-1} x_{t-1} u_{t-2}^{-1}, x_t u_{t-1}^{-1})\\ &= (e, \dots, e, x_t x_{t-1} \cdots x_2 x_1).\end{aligned}$$

(ii). Suppose that there exists $(a_1, \dots, a_t) \in L^t$ such that $(a_1, \dots, a_t) \rightharpoonup (e, \dots, e, \ell) = (e, \dots, e, k)$. Then $a_{t-1} = a_{t-2} = \dots = a_2 = a_1 = \theta(a_t)$, hence $k = a_t \ell \theta(a_t^{-1})$. Conversely, assume that $k = b\ell\theta(b^{-1})$; if $a_t = b$, $a_{t-1} = a_{t-2} = \dots = a_2 = a_1 = \theta(b)$, then $(a_1, \dots, a_t) \rightharpoonup (e, \dots, e, \ell) = (e, \dots, e, k)$. The second claim follows at once from the first and (i). Now (iii) is immediate.

(iv). Define the map $\psi : L^{t-1} \times \mathcal{O}_\ell^{L,\theta} \to \mathcal{C}_\ell$ by

$$\psi(b_1, \dots, b_{t-1}, c) = (b_1, \dots, b_{t-1}, \theta^{-1}(b_{t-1} \cdots b_1)^{-1} c).$$

It is a well-defined map by (ii) and it is clearly injective. Moreover, by the proof of (ii) we know that if $(b_1, \dots, b_t) \in \mathcal{C}_\ell$, then $b_t = \theta^{-1}(b_{t-1} \cdots b_1)^{-1} c$ for some $c \in \mathcal{O}_\ell^{L,\theta}$. Thus $(b_1, \dots, b_t) = \psi(b_1, \dots, b_{t-1}, c)$ by definition and ψ is surjective, implying that $|\mathcal{C}_\ell| = |L|^{t-1} |\mathcal{O}_\ell^{L,\theta}|$. $\square$

3.3. **Twisted homogeneous racks intersecting the diagonal.** Clearly,

$$G^u = \{(a, \dots, a) : a \in L^\theta\}, \tag{5}$$

hence, by Proposition 3.3

$$\mathcal{C}_\ell \cap G^u \neq \emptyset \iff \exists\, a \in L^\theta : a^t = \ell. \tag{6}$$

If this happens, then we have an inclusion of racks $\mathcal{O}_a^{L^\theta} \hookrightarrow \mathcal{C}_\ell$, see Remark 3.2. In the particular case $\theta = \mathrm{id}$, if there exists $a \in L$ such that $a^t = \ell$, we have an inclusion of racks $\mathcal{O}_a^L \hookrightarrow \mathcal{C}_\ell$.

3.4. Affine subracks of twisted homogeneous racks.

Lemma 3.4. *Let $\ell \in L$. Assume that there exists $x \in L$ such that*

$$\theta(\ell x \ell^{-1}) = x, \tag{7}$$

$$(k,t) \neq 1, \qquad \text{where } k = \operatorname{ord}(x), \tag{8}$$

$$k \neq 2, 4, \qquad \text{when } t = 2, \tag{9}$$

$$k > 2, \qquad \text{when } t = 4. \tag{10}$$

Then $\mathcal{C}_\ell$ is of type D.

When $\theta = \mathrm{id}$, (7) just means that $x \in \mathrm{C}_L(\ell)$.

Proof. Let $X = \{(x^{a_1}, x^{a_2}, \dots, x^{a_{t-1}}, \ell x^{a_t}) : \sum_{i=1}^t a_i \equiv 0 \mod k\}$. Then X is a subrack of $\mathcal{C}_\ell$ since $X \subseteq \mathcal{C}_\ell$, by Proposition 3.3 (ii), and

$$\begin{aligned}
(x^{a_1}, \dots, \ell x^{a_t}) \triangleright (x^{b_1}, \dots, \ell x^{b_t}) = \\
&= (x^{a_1}, \dots, \ell x^{a_t}) u((x^{b_1}, \dots, \ell x^{b_t})(x^{-a_1}, \dots, x^{-a_t}\ell^{-1})) \\
&= (x^{a_1}, \dots, \ell x^{a_t})(\theta(\ell x^{b_t - a_t}\ell^{-1}), \dots, x^{b_{t-1}-a_{t-1}}) \\
&= (x^{a_1 + b_t - a_t}, \dots, \ell x^{a_t + b_{t-1} - a_{t-1}}).
\end{aligned}$$

Moreover, let (A, g) be the affine rack considered in Lemma 2.1 or Lemma 2.2. Then by the calculation above, there exists a rack isomorphism

$$(A, g) \xrightarrow{\varphi} X, \qquad (a_1, \dots, a_{t-1}) \mapsto (x^{a_1}, \dots, x^{a_{t-1}}, \ell x^{-\sum_{i=1}^{t-1} a_i}),$$

which implies that X is of type D, by Lemma 2.1 or Lemma 2.2. $\square$

Corollary 3.5. *Assume that $t \geq 6$ is even. If $\ell \in L^\theta$, with $\operatorname{ord}(\ell)$ even, then $\mathcal{C}_\ell$ is of type D.*

Proof. Take $x = \ell$ and apply Lemma 3.4. $\square$

Lemma 3.6. *Assume $t = 2$ and $\theta = \mathrm{id}$. Let $\ell \in L$. Suppose that there exists $x, y, a \in L$ such that*

$$\begin{aligned}
&\ell = xy = yx, && x \neq e \neq y, && x^2 = e, \\
&a \in \mathrm{C}_L(x) \cap \mathrm{C}_L(y), && a^4 \neq e, && x \notin \langle a \rangle.
\end{aligned} \tag{11}$$

Then $\mathcal{C}_\ell$ is of type D.

Proof. Let $X = \{(xa^i, a^{-i}y) : \ 0 \le i < \operatorname{ord}(a)\}$ and $Y = \{(a^i, a^{-i}\ell) : \ 0 \le i < \operatorname{ord}(a)\}$. Both are subracks of $\mathcal{C}_\ell$ by Proposition 3.3 (ii) and the fact that $a \in \mathrm{C}_L(x) \cap \mathrm{C}_L(y)$. Since $x \notin \langle a \rangle$, we have that $X \cap Y = \emptyset$. Moreover, $X \coprod Y$ is a subrack of $\mathcal{C}_\ell$ since

$$(xa^i, a^{-i}y) \triangleright (a^j, a^{-j}\ell) = (a^{2i-j}, a^{-2i+j}\ell) \qquad \text{and}$$
$$(a^i, a^{-i}\ell) \triangleright (xa^j, a^{-j}y) = (xa^{2i-j}, a^{-2i+j}y).$$

Taking $r = (x, y)$ and $s = (a, a^{-1}\ell)$ we get that $r \triangleright (s \triangleright (r \triangleright s)) = (a^{-3}, a^3\ell)$. Since $a^4 \neq e$ by assumption, it follows that $\mathcal{C}_\ell$ is of type D. □

Corollary 3.7. *Assume $t = 2$ and $\theta = \mathrm{id}$. Let $\ell \in L$. If $\operatorname{ord}(\ell) = 2m$ with m odd, then $\mathcal{C}_\ell$ is of type D.*

Proof. The elements $x = \ell^m$, $y = a = \ell^{1-m}$ satisfy $\operatorname{ord}(y) = m$ and (11). □

Denote by $\mathbb{D}_n = \langle x, y \mid x^2 = 1 = y^n, \ xyx = y^{-1} \rangle$ the dihedral group of $2n$ elements.

Lemma 3.8. *Assume $t = 2$ and $\theta = \mathrm{id}$. If there exists $\psi : \mathbb{D}_n \to L$ a group monomorphism, with $n \ge 3$ odd, then $\mathcal{C}_{\psi(x)}$ is of type D.*

Proof. Let $z = \psi(y)$, $b_1 = \psi(x)$ and $b_j = b_1 z^{j-1}$, $2 \le j \le n$. Then $z^i b_j = b_{j-i}$ and $b_i z^j = b_{i+j}$, for all $1 \le i, j \le n$.

Let $R = \{(z^i, z^{-i}b_j) \mid 1 \le i, j \le n\}$ and $S = \{(z^{-k}b_l, z^k) \mid 1 \le k, l \le n\}$. They are disjoint since otherwise there exist i, k, $l \in \{1, \dots, n\}$ such that $z^i = z^{-k}b_l$; this implies that $z^{i+k} = b_l$, which is a contradiction because b_l is an involution and z has odd order. Note that $(z^i, z^{-i}b_j) = (z^i, b_{j+i})$ and $(z^{-k}b_l, z^k) = (b_{l+k}, z^k)$. Hence R and S are subracks of $\mathcal{C}_{\psi(x)}$ since

$$(z^i, b_{j+i}) \triangleright (z^k, b_{l+k}) = (z^{2i+j-k-l}, b_{k+j}),$$
$$(b_{j+i}, z^i) \triangleright (b_{l+k}, z^k) = (b_{k+j}, z^{2i+j-k-l}).$$

Moreover, $T = R \coprod S$ is a decomposable subrack of $\mathcal{C}_{\psi(x)}$ because

$$(z^i, b_{j+i}) \triangleright (b_{l+k}, z^k) = (b_{j-k}, z^{-(2i+j-k-l)}),$$
$$(b_{l+k}, x^k) \triangleright (z^i, b_{j+i}) = (z^{-(2k+l-j-i)}, b_{j-k}).$$

If we take $r = (1, b_1)$ and s$=(b_2, 1)$, then $r \triangleright (s \triangleright (r \triangleright s)) = (b_1b_2b_1, b_1b_2b_1b_2) \neq s$. Therefore, $\mathcal{C}_{\psi(x)}$ is of type D. □

3.5. **Twisted homogeneous racks with quasi-real ℓ.** Let $\ell \in L$ and $j \in \mathbb{N}$. Recall that ℓ, or $\mathcal{O}_\ell^L$, is quasi-real of type j if $\ell^j \neq \ell$ and $\ell^j \in \mathcal{O}_\ell^L$. For example, if ℓ is real, that is $\ell^{-1} \in \mathcal{O}_\ell^L$, but not an involution, then it is quasi-real of type $\operatorname{ord}(\ell) - 1$.

We partially extend this notion to twisted conjugacy classes. We shall say that $\ell \in L^\theta$ is *θ-quasi-real of type $j \in \mathbb{N}$* if $\ell^j \neq \ell$ and $\ell^j \in \mathcal{O}_\ell^{L,\theta}$. Note that if $\ell \in L^\theta$ is quasi-real of type j in L^θ, then it is θ-quasi-real, but the converse is not true. Indeed, if we take $L = \mathbb{A}_{11}$, $\theta = \iota_{(1\,2)}$ the conjugation by $(1\,2)$, $\ell = (1\ 2)(3\ 4)(5\ 6\ 7\ 8\ 9\ 10\ 11)$ and $y = (1\ 4)(2\ 3)(5\ 6\ 8)(7\ 10\ 9)$, then $y \rightharpoonup_\theta \ell = \ell^2$, but $\ell^2 \notin \mathcal{O}_\ell^{\mathbb{A}_{11}^\theta}$.

Lemma 3.9. *Assume that $\ell \in L^\theta$ is θ-quasi-real of type j. If $t \geq 3$ or $t = 2$ and $\operatorname{ord}(\ell) \nmid 2(1-j)$, then $\mathcal{C}_\ell$ is of type D.*

Proof. Let

$$X = \{(\ell^{a_1}, \dots, \ell^{a_t}) : \textstyle\sum_{i=1}^t a_i = j\},$$
$$Y = \{(\ell^{a_1}, \dots, \ell^{a_t}) : \textstyle\sum_{i=1}^t a_i = 1\}.$$

By definition, $X \cap Y = \emptyset$. We have

$$(\ell^{a_1}, \dots, \ell^{a_t}) \triangleright (\ell^{b_1}, \dots, \ell^{b_t}) = (\ell^{a_1+b_t-a_t}, \ell^{a_2+b_1-a_1}, \dots, \ell^{a_t+b_{t-1}-a_{t-1}}). \tag{12}$$

Hence X and Y are subracks of $\mathcal{C}_\ell$, $X \triangleright Y \subseteq Y$, $Y \triangleright X \subseteq X$, and $X \coprod Y$ is a subrack of $\mathcal{C}_\ell$.

Take $r = (1, \dots, 1, \ell^j) \in X$ and $s = (1, \dots, 1, \ell) \in Y$. We show that $r \triangleright (s \triangleright (r \triangleright s)) \neq s$. Assume first that $t \geq 4$. Then by (12) we have that

$$\begin{aligned} r \triangleright (s \triangleright (r \triangleright s)) &= r \triangleright (s \triangleright (\ell^{1-j}, 1, \dots, 1, \ell^j)) = r \triangleright (\ell^{j-1}, \ell^{1-j}, 1, \dots, 1, \ell) \\ &= (\ell^{1-j}, \ell^{j-1}, \ell^{1-j}, 1, \dots, 1, \ell^j) \neq (1, \dots, 1, \ell). \end{aligned}$$

If $t = 3$, then $r = (1, 1, \ell^j)$, $s = (1, 1, \ell)$ and the computation yields

$$\begin{aligned} r \triangleright (s \triangleright (r \triangleright s)) &= r \triangleright (s \triangleright (\ell^{1-j}, 1, \ell^j)) = r \triangleright (\ell^{j-1}, \ell^{-j+1}, \ell) \\ &= (\ell^{1-j}, \ell^{j-1}, \ell) \neq (1, 1, \ell). \end{aligned}$$

Finally if $t = 2$, then $r = (1, \ell^j)$, $s = (1, \ell)$ and we have

$$r \triangleright (s \triangleright (r \triangleright s)) = r \triangleright (s \triangleright (\ell^{1-j}, \ell^j)) = r \triangleright (\ell^{j-1}, \ell^{-j+2}) = (\ell^{-2j+2}, \ell^{2j-1}).$$

Thus, $\mathcal{C}_\ell$ is of type D if $(\ell^{-2j+2}, \ell^{2j-1}) \neq (1, \ell)$ which amounts to $\operatorname{ord}(\ell) \nmid 2(1-j)$. $\square$

4. Simple twisted homogeneous racks

Summarizing the results of the previous section, and with the same notation, we have

Proposition 4.1. *Let $\ell \in L^\theta$.*

(i) *If $\ell \in L$ is quasi-real of type j, $t \geq 3$ or $t = 2$ and $\operatorname{ord}(\ell) \nmid 2(1-j)$, then $\mathcal{C}_\ell$ is of type D.*
(ii) *If $\operatorname{ord}(\ell)$ is even and $t \geq 6$ is even, then $\mathcal{C}_\ell$ is of type D.*
(iii) *If ℓ is an involution, t is odd and $\mathcal{O}_\ell^{L^\theta}$ is of type D, then so is $\mathcal{C}_\ell$.*
(iv) *If ℓ is an involution, $t = 4$ and there exists $x \in C_{L^\theta}(\ell)$ with $\operatorname{ord}(x) = 2m > 2$, $m \in \mathbb{N}$, then $\mathcal{C}_\ell$ is of type D.*
(v) *If ℓ is an involution, $t = 2$, and there exists $x \in C_{L^\theta}(\ell)$ with $\operatorname{ord}(x) = 2m > 4$, $m \in \mathbb{N}$, then $\mathcal{C}_\ell$ is of type D.*
(vi) *If ℓ is an involution, $t = 2$, and there exists $\psi : \mathbb{D}_n \to L^\theta$ a group monomorphism, with $n \geq 3$ and $\ell = \psi(x)$ for some $x \in \mathbb{D}_n$ involution, then $\mathcal{C}_\ell$ is of type D.*
(vii) *If $(t, |L^\theta|)$ is divisible by an odd prime p, then $\mathcal{C}_e$ is of type D.*
(viii) *If $(t, |L^\theta|)$ is divisible by $p = 2$ and $t \geq 6$, then $\mathcal{C}_e$ is of type D.*
(ix) *If $t = 4$ and there exists $x \in L^\theta$ with $\operatorname{ord}(x) = 2m > 2$, $m \in \mathbb{N}$, then $\mathcal{C}_e$ is of type D.*
(x) *If $t = 2$, and there exists $x \in L^\theta$ with $\operatorname{ord}(x) = 2m > 4$, $m \in \mathbb{N}$, then $\mathcal{C}_e$ is of type D.*

Proof. (i) is Lemma 3.9; (ii) is Corollary 3.5; (iii) follows from the discussion in Subsection 3.3, since $(\ell, \dots, \ell) \in \mathcal{C}_\ell$. Case (vi) follows by Lemma 3.8. Finally, the remaining cases follow from Lemma 3.4: in cases (vii) and (viii), take x of order p, and in cases (iv), (v), (ix) and (x) the given x. □

We now explore when the proposition applies to a simple twisted homogeneous rack, that is when L is simple.

Remark 4.2. If L is simple non-abelian and $\theta = 1$, $t \neq 4, 2$ and $(t, |L|) \neq 1$, then $\mathcal{C}_e$ is of type D. This follows from Proposition 4.1 (vii) and (viii).

4.1. $L = \mathbb{A}_n$, $n \geq 5$, $\theta = \mathrm{id}$.

In this subsection we prove our main Theorem 1.2 in this case by applying all the results obtained above.

4.1.1. $\ell = e$. We now treat the Table 1.

- If $t \geq 6$ even or t odd with $(t, n!) \neq 1$, then $\mathcal{C}_e$ is of type D, by Lemma 3.4.

- Assume that $t = 4$. If $n \geq 6$, then $\mathcal{C}_e$ is of type D, by Proposition 4.1 (ix) with $x = (1\ 2)(3\ 4\ 5\ 6)$. If $n = 5$, we do not know if $\mathcal{C}_e$ is of type D.
- Assume that $t = 2$. If $n \geq 7$, then $\mathcal{C}_e$ is of type D, by Proposition 4.1 (x) with $x = (1\ 2)(3\ 4)(5\ 6\ 7)$. If $n = 5$ or 6, then Proposition 4.1 (x) does not apply because the only possible even orders of elements are 2 and 4. Moreover, we have checked that $\mathcal{C}_e$ is not of type D, using GAP.

4.1.2. *ℓ an involution.* The type of ℓ is $(1^{r_1}, 2^{r_2})$, with $n = r_1 + 2r_2$ and r_2 even. Assume that $\ell = (i_1\ i_2)(i_3\ i_4)\cdots(i_{2r_2-1}\ i_{2r_2})$. Then

- By [AFGV1, Thm. 4.1], $\mathcal{O}^L_\ell$ is of type D, except for the following cases:

$$(1, 2^2),\ n = 5; \qquad (1^2, 2^2),\ n = 6; \qquad (2^4),\ n = 8.$$

  In particular, $\mathcal{C}_\ell$ is of type D for all t odd, except for the cases above.
- If $t \geq 6$ is even, then $\mathcal{C}_\ell$ is of type D, by Proposition 4.1 (ii).
- Assume that $t = 4$. If $r_2 \geq 4$, then $\mathcal{C}_\ell$ is of type D by Proposition 4.1 (iv) with

$$x = (i_1\ i_2)(i_3\ i_5\ i_7\ i_4\ i_6\ i_8). \tag{13}$$

  Suppose that $r_2 = 2$. If $r_1 \geq 2$, then $\mathcal{C}_\ell$ is of type D by Proposition 4.1 (iv) with $x = (i_1\ i_3\ i_2\ i_4)(j_1\ j_2)$, where j_1, j_2 are fixed by ℓ. In the case $r_1 = 1$, Proposition 4.1 (iv) does not apply because the only possible even order of elements is 2.
- Assume that $t = 2$. If $r_2 \geq 4$, then $\mathcal{C}_\ell$ is of type D by Proposition 4.1 (v) with x as in (13). Suppose that $r_2 = 2$. If $r_1 \geq 3$, then $\mathcal{C}_\ell$ is of type D by Proposition 4.1 (v) with $x = (i_1\ i_2)(i_3\ i_4)(j_1\ j_2\ j_3)$, where j_1, j_2, j_3 are fixed by ℓ. In the case $r_1 = 1$ or 2, $\mathcal{C}_\ell$ yields of type D, by Lemma 3.8 taking $\psi : \mathbb{D}_3 \simeq \langle x := (1\ 2)(3\ 4), (1\ 2)(3\ 5)\rangle \hookrightarrow L$.

4.1.3. $\operatorname{ord}(\ell) > 2$. It is known that:

$$\begin{aligned} &\mathcal{O}^L_\ell \text{ is quasi-real of type 4, if } \operatorname{ord}(\ell) > 3 \text{ is odd,} \\ &\mathcal{O}^L_\ell \text{ is quasi-real of type } \operatorname{ord}(\ell) - 1, \text{ if } \operatorname{ord}(\ell) \text{ is even or 3.} \end{aligned} \tag{14}$$

Now, if $t > 2$ or $t = 2$ and $\operatorname{ord}(\ell) \neq 4$, then $\mathcal{C}_\ell$ is of type D, by Proposition 4.1 (i). On the other hand, if $t = 2$ and $\operatorname{ord}(\ell) = 4$, then Proposition 4.1 (i) does not apply and we do not know if $\mathcal{C}_\ell$ is of type D.

4.2. $L = \mathbb{A}_n$**,** $n \geq 5$**,** $\theta \neq \mathrm{id}$**.** We may suppose that θ is the inner automorphism associated with an element σ in $\mathbb{S}_n - \mathbb{A}_n$. We choose $\sigma = (1\ 2)$; thus, $\theta(\ell) = (1\ 2)\,\ell\,(1\ 2)$, for all $\ell \in \mathbb{A}_n$. Hence,

$$\mathbb{A}^\theta_n = \big(\{(1\ 2)\} \times (\mathbb{S}_{\{3,\dots,n\}} - \mathbb{A}_{\{3,\dots,n\}})\big) \coprod \mathbb{A}_{\{3,\dots,n\}} \tag{15}$$

and the order of L^θ is $(n-2)!$.

Let $\ell \in \mathbb{A}_n$. Recall that $\mathcal{C}_\kappa = \mathcal{C}_\ell$ if and only if $\kappa \in \mathcal{O}_\ell^{\mathbb{A}_n,\theta}$, by Proposition 3.3 (ii), and notice that $\kappa \in \mathcal{O}_\ell^{\mathbb{A}_n,\theta}$ amounts to $\kappa(1\ 2) \in \mathcal{O}_{\ell(1\ 2)}^{\mathbb{S}_n}$, the conjugacy class of $\ell(1\ 2)$ in $\mathbb{S}_n$. Thus, we will proceed with our study according to the type $(1^{s_1}, 2^{s_2}, \dots, n^{s_n})$ of $\ell(1\ 2)$.

Notice that $\operatorname{ord}(\ell(1\ 2))$ is even, since $\ell(1\ 2) \in \mathbb{S}_n - \mathbb{A}_n$, and $\operatorname{ord}(\ell(1\ 2)) \geq 4$ if and only if $s_h \geq 1$, for some $h \geq 4$, or $s_2 \geq 1$ and $s_3 \geq 1$.

Now, we want to give a description analogous to (14) for the case $\theta \neq \mathrm{id}$. First, we note that $\mathcal{O}_\ell^{\mathbb{A}_n,\theta} \cap \mathbb{A}_n^\theta \neq \emptyset$ if and only if $s_1 \geq 2$ or $s_2 \geq 1$. Indeed, this follows from the form of $\mathbb{A}_n^\theta$ described in (15).

Assume that $s_1 \geq 2$ or $s_2 \geq 1$ and let $\kappa \in \mathcal{O}_\ell^{\mathbb{A}_n,\theta} \cap \mathbb{A}_n^\theta$. Since $\mathcal{O}_\ell^{\mathbb{A}_n,\theta} = \mathcal{O}_{\ell(1\ 2)}^{\mathbb{S}_n} \cdot (1\ 2)$, it is clear that

(16) κ is θ-quasi-real of type j if and only if $\kappa^j(1\ 2) \in \mathcal{O}_{\ell(1\ 2)}^{\mathbb{S}_n}$.

If $\operatorname{ord}(\ell(1\ 2)) \geq 4$, then $\kappa(1\ 2)$ is quasi-real of type j in $\mathbb{S}_n$ with $j = \operatorname{ord}(\ell(1\ 2)) - 1$, by (14). Thus, $\kappa^j(1\ 2) = (\kappa(1\ 2))^j$ and $\kappa^j(1\ 2) \in \mathcal{O}_{\ell(1\ 2)}^{\mathbb{S}_n}$. Hence, κ is θ-quasi-real of type j, by (16); moreover, κ yields quasi-real of type j in $\mathbb{A}_n^\theta$. On the other hand, if $\operatorname{ord}(\ell(1\ 2)) = 2$, then $\kappa \in \mathcal{O}_{\ell(1\ 2)}^{\mathbb{S}_n}$ is θ-quasi-real (of type $j = 0$) if and only if $\mathcal{O}_\ell^{\mathbb{A}_n,\theta}$ contains to e, i. e. $\mathcal{O}_{\ell(1\ 2)}^{\mathbb{S}_n}$ is the conjugacy class of the transpositions in $\mathbb{S}_n$.

We study now the twisted homogeneous racks $(\mathbb{A}_n, t, \theta)$, $n \geq 5$ and $\theta \neq \mathrm{id}$, for different $t's$ according to the type $(1^{s_1}, 2^{s_2}, \dots, n^{s_n})$ of $\ell(1\ 2)$. If $s_1 \leq 1$ and $s_2 = 0$, then $\mathcal{O}_\ell^{\mathbb{A}_n,\theta} \cap \mathbb{A}_n^\theta = \emptyset$, and we do not know if $\mathcal{C}_\ell$ is of type D.

From now on we will assume that $s_1 \geq 2$ *or* $s_2 \geq 1$ *and let* $\kappa \in \mathcal{O}_\ell^{\mathbb{A}_n,\theta} \cap \mathbb{A}_n^\theta$. Notice that $\mathcal{C}_\ell = \mathcal{C}_\kappa$ and that $\ell(1\ 2)$ and $\kappa(1\ 2)$ have the same type, hence the same order. Moreover, if we denote the type of κ by $(1^{r_1}, 2^{r_2}, \dots, n^{r_n})$, then $r_h = s_h$, for all h, $3 \leq h \leq n$; thus, $\operatorname{ord}(\kappa)$ divides $\operatorname{ord}(\ell(1\ 2))$.

We will consider different cases. If $s_h \geq 1$ for $h = 3$ or $5 \leq h \leq n$, then $\operatorname{ord}(\ell(1\ 2)) > 4$ and $\mathcal{C}_\ell$ is of type D for all t, by Proposition 4.1 (i). Assume that $s_h = 0$ for $h = 3$ and $5 \leq h \leq n$. Suppose that $s_4 \geq 1$; thus $\operatorname{ord}(\ell(1\ 2)) = 4$. If $t \geq 3$, then $\mathcal{C}_\ell$ is of type D for all t, by Proposition 4.1 (i); whereas if $t = 2$, we do not know if $\mathcal{C}_\ell$ is of type D. Assume that $s_4 = 0$. Thus, the type of $\ell(1\ 2)$ is $(1^{s_1}, 2^{s_2})$, with $s_1 \geq 2$ or $s_2 \geq 1$ odd.

Suppose that $s_2 = 1$; thus, $e \in \mathcal{O}_\ell^{\mathbb{A}_n,\theta}$. If $t \geq 3$, then $\mathcal{C}_\ell = \mathcal{C}_e$ is of type D for all t, by Proposition 4.1 (i); whereas if $t = 2$, $n \geq 7$ and take $\kappa = e$, then $\mathcal{C}_\ell = \mathcal{C}_e$ is of type D, by Proposition 4.1 (x) choosing $x = (1\ 2)(3\ 4)(5\ 6\ 7)$.

Suppose that $s_2 > 1$ odd. Then the type of $\kappa \in \mathcal{O}_\ell^{\mathbb{A}_n,\theta} \cap \mathbb{A}_n^\theta$ is $(1^{r_1}, 2^{r_2})$, with $n = r_1 + 2r_2$ and $r_2 \geq 2$ even, i. e. $\kappa = (i_1\ i_2)(i_3\ i_4)\cdots(i_{2r_2-1}\ i_{2r_2})$. We determine now when $\mathcal{O}_\kappa^{\mathbb{A}_n^\theta}$ is of type D. We have two possibilities.

(i) Assume that κ fixes 1 and 2. If the type $(1^{r_1-2}, 2^{r_2})$ is distinct to $(1, 2^2)$, $(1^2, 2^2)$ and (2^4), then $\mathcal{O}_\kappa^{\mathbb{A}_n^\theta}$ is of type D, by [AFGV1, Thm. 4.1]; otherwise, $\mathcal{O}_\kappa^{\mathbb{A}_n^\theta}$ is not of type D.

(ii) Assume that κ does not fix 1 nor 2; thus $\kappa = (12)(i_3 i_4)\cdots(i_{2r_2-1} i_{2r_2})$. If the type $(1^{r_1}, 2^{r_2-1})$ is distinct to (2^3) and $(1^{r_1}, 2)$, then $\mathcal{O}_\kappa^{\mathbb{A}_n^\theta}$ is of type D, by [AFGV1, Thm. 4.1]; otherwise, $\mathcal{O}_\kappa^{\mathbb{A}_n^\theta}$ is not of type D.

We consider now different values of t.

- Assume that t is odd. If κ fixes 1 and 2 and the type of κ is distinct to $(1^3, 2^2)$, $(1^4, 2^2)$ and $(1^2, 2^4)$, then $\mathcal{C}_\ell$ is of type D, by (i) above and Proposition 4.1 (iii). On the other hand, if κ does not fix 1 nor 2 and the type of κ is distinct to (2^4) and $(1^{r_1}, 2^2)$, for any r_1, then $\mathcal{C}_\ell$ is of type D, by (ii) above and Proposition 4.1 (iii).
- If $t \geq 6$ even, then $\mathcal{C}_\ell$ is of type D, by Proposition 4.1 (ii).
- Assume that $t = 4$. We will determine when there exists $x \in \mathbb{A}_n^\theta$ such that $\operatorname{ord}(x) \geq 4$ even and $\theta(\kappa x \kappa) = x$, i. e. $(1\ 2)\kappa x \kappa (1\ 2) = x$. If κ fixes 1 and 2, take $x = (1\ 2)(i_1\ i_3\ i_2\ i_4)$. If $\kappa(1) = 2$ and $\kappa(2) = 1$, take $x = (1\ 2)(i_3\ i_5\ i_4\ i_6)$ when $r_2 \geq 4$ and $x = (1\ j_1\ 2\ j_2)(i_3\ i_4)$ when $r_2 = 2$ and $r_1 \geq 2$, j_1, j_2 are fixed by κ. In all these cases, $\mathcal{C}_\ell$ is of type D, by Lemma 3.4. For the remaining cases we do not know if $\mathcal{C}_\ell$ is of type D.
- Assume that $t = 2$. We will determine when there exists $x \in \mathbb{A}_n^\theta$ such that $\operatorname{ord}(x) \geq 6$ even and $\theta(\kappa x \kappa) = x$, i. e. $(1\ 2)\kappa x \kappa (1\ 2) = x$. If κ fixes 1 and 2, take $x = (1\ 2)(i_3\ i_5\ i_7\ i_4\ i_6\ i_8)$ when $r_2 \geq 4$ and $x = (1\ i_1\ i_3\ 2\ i_2\ i_4)(j_1\ j_2)$ when $r_2 = 2$ and $r_1 \geq 4$, j_1, j_2 are fixed by κ. If $\kappa(1) = 2$ and $\kappa(2) = 1$, take $x = (1\ 2)(i_3\ i_5\ i_7\ i_4\ i_6\ i_8)$ when $r_2 \geq 4$ and $x = (1\ 2)(i_3\ i_4)(j_1\ j_2\ j_3)$ when $r_2 = 2$ and $r_1 \geq 3$, j_1, j_2, j_3 are fixed by κ.

Therefore the cases where $\mathcal{C}_\ell$ is not known to be of type D are the following

(a) $\ell(1\ 2)$ of type $(1^{s_1}, 2^{s_2}, \dots, n^{s_n})$, $s_1 \leq 1$ and $s_2 = 0$ for any t;
(b) $\ell(1\ 2)$ of type $(1^{s_1}, 2^{s_2}, 4^{s_4})$, $s_1 \leq 2$ or $s_2 \geq 1$, $s_2 + s_4$ odd, $s_4 \geq 1$ and $t = 2$;

and those in Table 4. Finally, from (a), (b) and Table 4 we obtain Table 2.

TABLE 4.

n	Type of ℓ	ℓ	t
6	$(1^2, 2^2)$	involution	2
7	$(1^3, 2^2)$	fixing 1 and 2	2, odd
8	$(1^4, 2^2)$		odd
10	$(1^2, 2^4)$		odd
5	$(1, 2^2)$	involution	2, 4
6	$(1^2, 2^2)$	permuting 1 and 2	2
8	(2^4)		odd

4.3. $L =$ **sporadic group,** $\theta = \mathrm{id}$**.** In this subsection we prove our main Theorem 1.3.

4.3.1. $\ell = e$.

- If $(t, |L|) \neq 1$, with t odd or $t \geq 6$ even, then $\mathcal{C}_e$ is of type D; see Table 5 for the prime numbers dividing the order of a sporadic group. In particular, if $t \geq 6$ even, then $\mathcal{C}_e$ is of type D since $|L|$ is even.

TABLE 5. Prime divisors of orders of sporadic groups.

L	**Prime divisors**	L	**Prime divisors**
M_{11}, M_{12}	2, 3, 5, 11	Co_1	2, 3, 5, 7, 11, 13, 23
M_{22}, HS, McL	2, 3, 5, 7, 11	J_1, HN	2, 3, 5, 7, 11, 19
M_{23}, M_{24}, Co_2, Co_3	2, 3, 5, 7, 11, 23	$O'N$	2, 3, 5, 7, 11, 19, 31
J_2	2, 3, 5, 7	J_3	2, 3, 5, 17, 19
Suz, Fi_{22}	2, 3, 5, 7, 11, 13	Ru	2, 3, 5, 7, 13, 29
T	2, 3, 5, 13	Fi_{23}	2, 3, 5, 7, 11, 13, 17, 23
He	2, 3, 5, 7, 17	Fi_{24}'	2, 3, 5, 7, 11, 13, 17, 23, 29
Th	2, 3, 5, 7, 13, 19, 31	B	2, 3, 5, 7, 11, 13, 17, 19, 23, 31, 47
J_4	2, 3, 5, 7, 11, 23, 29, 31, 37, 43	M	2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71
Ly	2, 3, 5, 7, 11, 31, 37, 67		

- If $t = 2$ or $t = 4$, then $\mathcal{C}_e$ is of type D, by Proposition 4.1 (ix) and (x), since always there exists an element $x \in L$ of order 6.

4.3.2. ℓ *an involution.*

- By [AFGV2, Thm. II], if ℓ is an involution then, $\mathcal{O}_\ell^L$ is of type D, except for the cases listed in Table 6; in particular, $\mathcal{C}_\ell$ is of type D for all t odd, except for these cases.

TABLE 6. Classes of involutions not known of type D;
those which are NOT of type D appear in bold.

G	**Classes**	G	**Classes**	G	**Classes**
J_2	**2A**	Fi_{22}	**2A**	Co_2	**2A**
B	2A, 2C	Fi_{23}	**2A**	T	2A

- If $t \geq 6$ is even, then $\mathcal{C}_\ell$ is of type D, by Proposition 4.1 (ii).
- If $t = 2$ or 4, then $\mathcal{C}_\ell$ is of type D by Proposition 4.1 (iv) and (v), since always there exists $x \in C_L(\ell)$ with $\mathrm{ord}(x) > 4$ even. To see this we use [BR, Bo, Br, GAP, Iv, W, W+].

4.3.3. $\mathrm{ord}(\ell) > 2$. The class $\mathcal{O}_\ell^L$ is real or quasi-real, except the classes 6B, 6C of the Suzuki group Suz.

- Assume that ℓ do not belong to the class 6B or 6C of the Suzuki group Suz. If $t \geq 3$ or $t = 2$ and $\mathrm{ord}(\ell) \neq 4$, then $\mathcal{C}_\ell$ is of type D, by Proposition 4.1 (i).
- Suppose that ℓ belongs to the class 6B or 6C of the Suzuki group Suz. If $t \geq 6$ even, then $\mathcal{C}_\ell$ is of type D by Proposition 4.1 (ii); whereas if $t = 2$, then $\mathcal{C}_\ell$ is of type D by Corollary 3.7.

4.4. L = **sporadic,** $\theta \neq \mathrm{id}$**.** The sporadic groups with non-trivial outer automorphism group are M_{12}, M_{22}, J_2, Suz, HS, McL, He, Fi_{22}, Fi_{24}', $O'N$, J_3, T and HN. For these groups the outer automorphism group is $\mathbb{Z}/2$ in all cases. In Table 7 we give the orders of L^θ when $L \neq HN$; we cannot determine the order of HN^θ with our computational resources. We will assume $L \neq HN$.

Table 7. Orders of L^θ.

L	$\lvert L^\theta\rvert$	L	$\lvert L^\theta\rvert$	L	$\lvert L^\theta\rvert$
M_{12}	120	M_{22}	1344	J_2	336
Suz	1209600	HS	40320	McL	7920
He	7560	Fi_{22}	54	Fi_{24}'	4089470473293004800
T	96	$O'N$	175560	J_3	2448

We describe the case $\ell = e$. The remaining cases will be treated in a separated paper.

- If $t \geq 6$ even or $t \geq 3$ odd and $(t, |L^\theta|) \neq 1$, then $\mathcal{C}_e$ is of type D, by Proposition 4.1 (vii) and (viii). In particular, if $t \geq 6$ even, then $\mathcal{C}_e$ is of type D since $|L^\theta|$ is even.
- Assume that $t = 2$ or $t = 4$. We have checked with GAP that there exists $x \in L^\theta$ such that $\mathrm{ord}(x) > 4$ even. Then $\mathcal{C}_e$ is of type D, by Proposition 4.1 (ix) and (x).

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